Raman scattering and excitation of the harmonics of intense laser radiation in cold plasmas

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Plane linearly and circularly polarized intense electromagnetic waves in cold underdense plasmas are considered and approximate expressions for them and adiabatic relations between their local amplitudes and frequencies are established. A general three-dimensional theory is developed for the instability of propagation of a plane monochromatic circularly polarized electromagnetic wave in plasma including the violation of its initial polarization. Excitation of harmonics resulting from the relativistic and charge-displacement nonlinearities, scattering due to the response of the electron fluid, decay instability of harmonics leading to the emergence of scattered electromagnetic waves and plasmons, wave-wave interactions, and the generation of a continuum of scattered radiation are studied. [S1063-651X(99)10301-5]

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I. INTRODUCTION AND BASIC EQUATIONS

A number of recent works were dedicated to theoretical and experimental studies of the propagation of ultrahigh intensity laser radiation in matter [1–17]. Intensities on the order of 10^{18} W/cm^2 or higher are considered ultrahigh since the motions of electrons driven by them are relativistic. At present ultrashort laser pulses of such intensities are used in experiments [14–16], including experimental observations of scattering [17]. The central part of an ultrashort laser pulse focused into matter interacts with the plasma that is formed at its front. It is well known that matter irradiated by a powerful laser can be polarized due to nonlinear currents of free electrons [18], deformations of electron shells of atoms and ions [19], and vibrations and rotations of molecules [20]. The first of the above effects plays a major role in experiments with light atomic gases since in this case the ionization of matter is complete. Below we consider the scattering of laser radiation under these circumstances. Scattering of laser pulses in plasmas at nonrelativistic intensities was studied, for example, in [18,21–26]. Scattering and excitation of harmonics at relativistic intensities were treated in papers [1,2,9–11,13] (see also [3–7]).

Schematically one can say that there are two practically unrelated aspects of the problem of scattering of laser radiation in matter. The first aspect concerns finding local characteristics of the medium, namely, the temporal growth rates (spatial gains) of the scattered radiation in a plasma infinitesimal volume as functions of components of the scattered light wave vector and the incident pulse parameters. In contrast, the second one is an integrodifferential transport problem of calculating the radiation field far from the scattering domain accounting for absorption and gain on the total propagation distance. The first of the above problems is treated in detail in the present paper: the scattering temporal growth rates are calculated. The second problem is considered on a qualitative level only.

The task of developing a spatially three-dimensional model of scattering of circularly polarized relativistically intense plane monochromatic electromagnetic waves in plasma was addressed recently [10,13]. The theory proposed in these works includes a range of wave phenomena: excitation of harmonics, Raman scattering, the fluid dynamical analog of Compton-effect, etc., as well as the limits that have been studied previously, mainly the nonrelativistic one. Reference [11] is dedicated to the same problem but a linearly polarized pump is considered in it.

However, various approximations are used in the existing literature due to the complexity of the problem of scattering of relativistically intense electromagnetic waves in plasma. These approximations include (1) the slab geometry limit (see, for example, [9]); (2) the assumption of conservation of a certain type of polarization, circular in particular [13]; (3) the calculation of growth rates under the assumption that one of the wave vector components is zero [10]; and (4) resonance conditions based on exact phase matching (see, for example, [24]).

Neither of the above approximations is used in the framework of the theory of scattering of relativistically intense laser radiation in plasmas, which is developed in the present

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paper. We employ numerical techniques to describe harmonics excitation, Raman scattering by plasmons, the fluid dynamics analog of Compton scattering, generation of a continuum of radiation, and the interplay of all these phenomena. Linearized Maxwell equations and the equations of fluid dynamics for the electron component of laser-irradiated plasma are analyzed rigorously.

One needs an exact ground-state solution of the initial nonlinear set of equations to perform such analyses. In general, plane electromagnetic waves of arbitrarily high intensities propagating in cold underdense plasmas are described by solutions to the classic Akhiezer-Polovin problem [27]. Below we develop a class of approximate solutions to this problem corresponding to high frequency electromagnetic waves with the help of a nonlinear analog of the WKB approximation. In the present paper the exact solution corresponding to a plane circularly polarized monochromatic electromagnetic wave [27] is used for the linear instability analysis.

It should be noted that stability analysis was performed a number of times for linearly polarized monochromatic waves that do not correspond to exact solutions of the initial nonlinear set of equations and consequently the resulting theory was applicable to the case of low intensities only.

The problem of instability of a plane circularly polarized monochromatic electromagnetic wave in plasma is reduced to a set of linear partial differential equations with rapidly oscillating coefficients. When a comoving variable is introduced and the equations are Fourier transformed in space an infinite linear set of coupled ordinary differential equations is derived (the infinite dimension of the set is related to the need to describe excitation of numerous harmonics and their interactions). The simulations performed show that a correct solution to the above problem can be obtained by treating over a hundred coupled ordinary differential equations. The temporal growth rate of the problem is defined as the maximal eigenvalue of the matrix of the above set of equations. In particular, this approach makes it possible to avoid deriving and solving complicated dispersion relations. Note that a similar technique is applied in fluid dynamics for analyzing the linear stage of the emergence of turbulence [28]. Previously the authors of the present paper used this method to analyze the instability of a plane electromagnetic wave in the framework of the circular polarization conservation approximation [13].

Thus, the results of a rigorous linear analysis of the set of Maxwell equations and the relativistic fluid dynamics of plasma electrons are presented below for the first time.

The propagation of relativistically intense laser radiation in cold underdense plasma is described by the Maxwell equations and the equations of relativistic fluid dynamics for the plasma electron component (for example, see [29]),

$$(\Delta-\partial_t^2)\mathbf{A}=\nabla\varphi_t+\frac{n}{\gamma}(\mathbf{A}+\nabla\psi), \tag{1.1}$$

$$\Delta\varphi=n-1, \tag{1.2}$$

$$(\nabla,\mathbf{A})=0, \tag{1.3}$$

$$\psi_t=\varphi-\gamma, \tag{1.4}$$

$$n_t+\left(\nabla,\frac{n}{\gamma}(\mathbf{A}+\nabla\psi)\right)=0, \tag{1.5}$$

$$\gamma=\sqrt{1+|\mathbf{A}+\nabla\psi|^2}. \tag{1.6}$$

Here $\mathbf{A}$ and φ are the vector and scalar potentials of the electromagnetic field, ψ is the potential of the generalized momentum, and n is the electronic concentration. Equation (1.6) defines the relativistic mass factor γ.

The quantities in Eq. (1.1)–(1.6) are normalized as follows: $\mathbf{A}$ and φ are normalized by mc^2/e, n—by its unperturbed value n_0, time—by $1/\omega_p$, where ω_p is the unperturbed plasma frequency, and the coordinates are normalized by c/ω_p.

Conservation laws for the system comprising Eqs. (1.1)–(1.6) can be found, for example, in [29].

II. SELF-MODULATED RELATIVISTICALLY INTENSE HIGH FREQUENCY PLANE ELECTROMAGNETIC WAVES IN COLD UNDERDENSE PLASMAS

A. Akhiezer-Polovin problem

The propagation of relativistically intense plane electromagnetic waves in cold plasmas is described by the classic Akhiezer-Polovin problem [27]. In general the solutions to this problem can be developed numerically. Analytical expressions for these solutions are available for low amplitude and purely longitudinal waves [27] as well as when the longitudinal component of the plasma electronic momentum is small [30,31]. A numerical study of solutions to the Akhiezer-Polovin problem on the phase plane is presented in [32].

In the present section we develop some numerical and approximate analytical solutions to the Akhiezer-Polovin problem. These solutions describe linearly and circularly polarized self-modulated electromagnetic waves propagating in plasma with a phase velocity close to the speed of light.

Following [27] we put $\nabla_\perp=0$ in Eq. (1.1)–(1.6) and further assume that all the unknown functions depend on the comoving variable $\xi=x-qt$, where the phase velocity (normalized by the speed of light) is expressed as $q=(1+\varepsilon^2)^{1/2}$, ε being the problem parameter. As shown in [27], in this case the problem is reduced to the following set of coupled nonlinear ordinary differential equation for the vector and scalar potentials

$$\varepsilon^2\mathbf{A}_{\xi\xi}+F(\mathbf{A},\varphi,\varepsilon)\mathbf{A}=\mathbf{0}, \tag{2.1}$$

$$\varepsilon^2\varphi_{\xi\xi}+F(\mathbf{A},\varphi,\varepsilon)\varphi-1=0, \tag{2.2}$$

$$F(\mathbf{A},\varphi,\varepsilon)=\sqrt{\frac{1+\varepsilon^2}{\varphi^2+\varepsilon^2(1+|\mathbf{A}|^2)}}. \tag{2.3}$$

The invariants of this problem are expressed as [27]

$$\varepsilon^2|\mathbf{A}_\xi|^2+\varphi_\xi^2+W(\mathbf{A},\varphi,\varepsilon)=E\equiv\text{const},$$

$$W(\mathbf{A},\varphi,\varepsilon)=\frac{2}{\varepsilon^2}\left(\sqrt{1+\varepsilon^2}\sqrt{\varphi^2+\varepsilon^2(1+|\mathbf{A}|^2)}-\varphi\right),$$

$$A_1A_{2_\xi}-A_2A_{1_\xi}=M\equiv\text{const},$$

where A_1 and A_2 are the transverse components of the vector potential ($A_3=0$ in the Coulomb gauge). The longitudinal

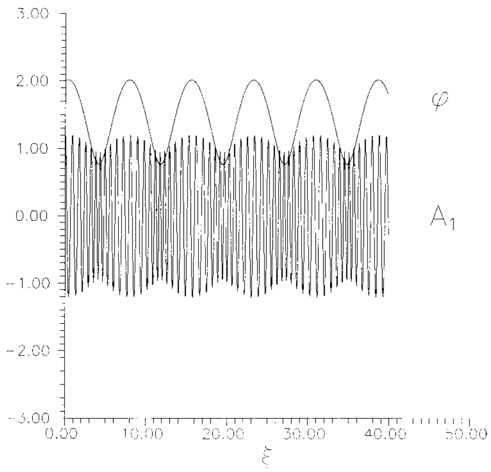

FIG. 1. Numerical solutions to the Akhiezer-Polovin problem [Eqs. (2.1)–(2.3)] for the initial conditions: $A_1(0)=1.2$, $A_{1\xi}(0)=0$, $\varphi(0)=2$, $\varphi_\xi(0)=0$, $\epsilon=0.1$.

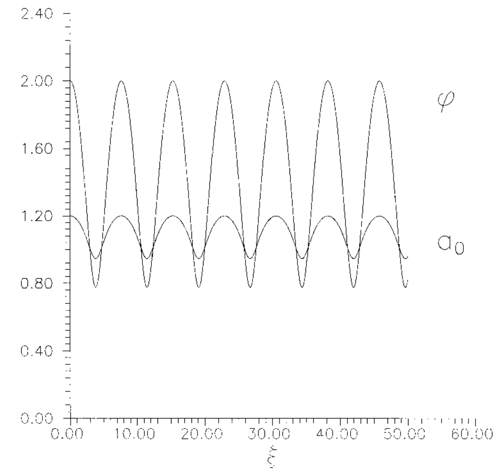

FIG. 2. The scalar potential φ and the amplitude of the vector potential calculated using Eqs. (2.9) and (2.11). The initial conditions are the same as in Fig. 1.

momentum of the plasma electron component and the perturbation of the electron concentration are functions of the vector and scalar potentials

$$p=\varepsilon^{-2}(\sqrt{\varphi^2+\varepsilon^2(1+|\mathbf{A}|^2)}-\varphi\sqrt{1+\varepsilon^2}), \tag{2.4}$$

$$n-1=\varepsilon^{-2}\left(1-\frac{\varphi\sqrt{1+\varepsilon^2}}{\sqrt{\varphi^2+\varepsilon^2(1+|\mathbf{A}|^2)}}\right). \tag{2.5}$$

B. Linearly polarized plane electromagnetic waves

Figure 1 shows a numerical solution to Eqs. (2.1)–(2.3) (linear polarization). Obviously, amplitude modulation occurs, namely, the electromagnetic radiation is concentrated between the peaks of the electrostatic potential. One can also see that the frequency of the vector potential oscillation varies between the peak and the minimum of the scalar potential, i.e., phase modulation also takes place.

Generally the phase velocity of electromagnetic waves is close to the speed of light and the parameter ε is small in case the frequency of the propagating laser pulse is much greater than the unperturbed plasma frequency. Our goal is to develop asymptotic expansions of the solutions to Eqs. (2.1)–(2.3) in ε in this particular case. We use the ansatz (see, for example, [33])

$$A_1(\xi)=U(\xi,\Theta)+\sum_{m=1}^{\infty}\varepsilon^m U_m(\xi,\Theta), \tag{2.6}$$

$$\varphi(\xi)=\phi(\xi,\Theta)+\sum_{m=1}^{\infty}\varepsilon^m \phi_m(\xi,\Theta), \tag{2.7}$$

$$\Theta_\xi=\varepsilon^{-1}k(\xi). \tag{2.8}$$

Here $k(\xi)$ is an additional independent function. Substituting Eqs. (2.6)–(2.8) in Eqs. (2.1)–(2.3) and eliminating secular terms in the lowest orders in ε we arrive at the following averaged nonlinear equation:

$$\phi_{\xi\xi}=\frac{1}{2}\left(\frac{1+\frac{\alpha^2}{2}\phi^{1/2}}{\phi^2}-1\right), \tag{2.9}$$

and find that $k(\xi)=\phi^{-1/2}(\xi)$ and

$$\Theta=\varepsilon^{-1}\int\phi^{-1/2}d\xi, \tag{2.10}$$

$$U(\xi,\Theta)=\alpha\phi^{1/4}\sin\Theta. \tag{2.11}$$

In the above equation α can be interpreted as the parameter of coupling of the electromagnetic field and the Langmuir wake of the plasma. The value of this parameter is determined by initial conditions. The following conservation law is associated with Eq. (2.9):

$$\phi_\xi^2+V(\phi)=E=\text{const}, \tag{2.12}$$

$$V(\phi)=\phi+\phi^{-1}+\alpha^2\phi^{-1/2}. \tag{2.13}$$

Figure 2 shows the scalar potential φ and the self-modulated amplitude of the vector potential calculated using Eqs. (2.9) and (2.11). The initial conditions are the same as in Fig. 1.

Note that a similar technique was used in [34] to describe soliton modes of propagation of relativistically intense laser pulses in plasmas.

It follows easily from Eqs. (2.12) and (2.13) that the period of the slow Langmurian waves generated by the propagating rapidly oscillating field is given by

$$T(E,g^2)=\oint\frac{d\phi}{\sqrt{E-V(\phi)}}, \tag{2.14}$$

where $\oint$ denotes integration over a full cycle between the two solutions of $V(\phi)=E$.

Combining Eqs. (2.6)–(2.8), (2.11), (2.4), and (2.5) one finds that the concentration and longitudinal momentum of the plasma electron component are

$$n-1=\frac{1}{2}\left(\frac{1+\frac{\alpha^2}{2}\,\phi^{1/2}}{\phi^2}-1\right)-\frac{\alpha^2}{4\phi^{3/2}}\cos 2\Theta,$$

$$p=\phi(n-1), \tag{2.15}$$

which shows that the electron fluid concentration and momentum exhibit a slow response to the propagating electromagnetic field and oscillations at the frequency of the second harmonics.

Two simple analogies are related to Eqs. (2.10) and (2.11). Note that a formal expansion of the nonlinearity in Eq. (2.1) in ε^2 results in

$$\varepsilon^2\mathbf{A}_{\xi\xi}+\varphi^{-1}\mathbf{A}=\mathbf{0}.$$

First, if we assume that φ varies slowly [and this assumption is natural since expanding Eq. (2.2) in ε^2 formally one obtains an equation that is not singularly perturbed] the above equation can be interpreted as the Einstein pendulum (i.e., a pendulum with a slowly varying frequency) problem [35]. Even though in our case this frequency is not an explicitly defined function but must be calculated from another nonlinear equation there is the same relation between the local frequency

$$\Omega(\xi)=\phi^{-1/2}(\xi)/\varepsilon$$

and amplitude $a_0=\alpha\phi^{1/4}(\xi)$: $a_0^2\Omega=\alpha^2/\varepsilon$. Consequently the constant α^2/ε can be interpreted as an adiabatic invariant of problem (2.1)–(2.3). It should be noted that since the frequency shift can be recorded experimentally the present amplitude-frequency relation can be used for experimental diagnostics of the propagation process.

Second, the above equation for the vector potential is formally analogous to the Schrödinger equation for a particle in a potential well. In the framework of this analogy the small parameter ε and the function φ^{-1} play the roles of the Plank constant and the square of the particle classical momentum, respectively. Obviously the theory derived above is similar to the WKB approximation.

The quantization condition for Eqs. (2.9)–(2.13) is also analogous to that of quantum mechanics,

$$J=\oint\phi^{-1/2}d\xi=\oint\frac{d\phi}{\sqrt{\phi(E-V(\phi))}}=2\pi\varepsilon m,$$

where $m\gg 1$ is an integer. The last equation is the dispersion relation for the plane waves studied since the local frequency of the electromagnetic field averaged over the slow period makes $\overline{\Omega}=(J/\varepsilon T)=(2\pi m/T)$. The quantization conditions correspond to the periodic solutions of the Akhiezer-Polovin problem.

The above quantization condition can be used to find the average intensity of the electromagnetic wave defined by a solution to the Akhiezer-Polovin problem. It can be derived easily that the corresponding normalized intensity is

$$I=\frac{\alpha^2}{\varepsilon^2\phi^{1/2}}\cos^2\Theta.$$

Averaging it first over the fast oscillations and then over the slow wave period T and using the quantization condition we get

$$\overline{I}=\frac{\pi m\alpha^2}{\varepsilon T}.$$

The simplest solution to Eq. (2.9) is a constant $\phi=\phi_0>1$, the corresponding electromagnetic field being monochromatic,

$$U=\sqrt{2(\phi_0^2-1)}\sin\frac{\xi}{\varepsilon\sqrt{\phi_0}}.$$

The linear instability of this solution is studied in [11].

C. Circularly polarized plane electromagnetic waves

Asymptotic solutions to problems (2.1)–(2.3) corresponding to the case of circular polarization are derived in a manner similar to that outlined above and expressed as

$$A_1(\xi)=\alpha\phi^{1/4}(\xi)\cos\Theta,$$

$$A_2(\xi)=\pm\alpha\phi^{1/4}(\xi)\sin\Theta,$$

and the equation for ϕ and its first integral differ from those given by Eqs. (2.9), (2.12), and (2.13) in the following way: there is α^2 instead of $\alpha^2/2$ and $2\alpha^2$ instead of α^2 in them, respectively. In this case there is no second harmonics component in the plasma wake,

$$n-1=\frac{1}{2}\left(\frac{1+\alpha^2\phi^{1/2}}{\phi^2}-1\right)$$

[the electron fluid momentum is related to the above quantity by Eq. (2.15) just as in the case of linear polarization].

Also, just like the case of linear polarization there exist monochromatic circularly polarized waves given by [27]

$$A_1=\sqrt{\phi_0^2-1}\,\cos\frac{\xi}{\varepsilon\sqrt{\phi_0}}, \tag{2.16}$$

$$A_2=\pm\sqrt{\phi_0^2-1}\,\sin\frac{\xi}{\varepsilon\sqrt{\phi_0}}. \tag{2.17}$$

It should be noted that this particular solution is an exact one. A number of works was dedicated to its instability (for example, see [9–13]), which will be considered below as well.

III. EQUATIONS DESCRIBING THE INSTABILITY OF CIRCULARLY POLARIZED MONOCHROMATIC LASER RADIATION IN PLASMAS

Introducing the notations

$$a_0=\sqrt{\phi_0^2-1},\quad \phi_0=\gamma_0,\quad k=\frac{1}{\varepsilon\sqrt{\phi_0}},$$

we present Eqs. (2.16) and (2.17) describing the propagation of a plane monochromatic circularly polarized wave in the form

$$\mathbf{A}_0=\tfrac{1}{2}(\mathbf{e}_1+i\mathbf{e}_2)a_0\exp(ik\xi)+\text{c.c.} \tag{3.1}$$

In this case we also have

$$\omega^2-k^2=\gamma_0^{-1},\quad \gamma_0=\sqrt{1+a_0^2},\quad n=1,\quad \varphi=\gamma_0,\quad \psi=0.$$

Consider the evolution of small perturbations (which are marked by $\sim$) in plasma against the background defined by the above relations,

$$\mathbf{A}=\mathbf{A}_0+\tilde{\mathbf{A}},\quad n=1+\tilde{n},\quad \varphi=\gamma_0+\tilde{\varphi},\quad \psi=\tilde{\psi}. \quad (3.2)$$

The linearized equations governing the perturbations are

$$(\Delta-\partial_t^2)\tilde{\mathbf{A}}=\nabla\tilde{\varphi}_t+\gamma_0^{-1}(\tilde{\mathbf{A}}+\nabla\tilde{\psi}+\tilde{n}\mathbf{A}_0 - \gamma_0^{-3}(\mathbf{A}_0,\tilde{\mathbf{A}}+\nabla\tilde{\psi})\mathbf{A}_0, \quad (3.3)$$

$$\Delta\tilde{\varphi}=\tilde{n}, \quad (3.4)$$

$$(\nabla,\tilde{\mathbf{A}})=0, \quad (3.5)$$

$$\tilde{\psi}_t=\tilde{\varphi}-\gamma_0^{-1}(\mathbf{A}_0,\tilde{\mathbf{A}}+\nabla\tilde{\psi}). \quad (3.6)$$

The following continuity equation is obtained by calculating the divergence of the left and right hand sides of Eq. (3.3) and using Eq. (3.4),

$$\tilde{n}_t+\gamma_0^{-1}(\mathbf{A}_0,\nabla\tilde{n}) = -\gamma_0^{-1}\Delta\tilde{\psi}+\gamma_0^{-3}(\mathbf{A}_0,\nabla(\mathbf{A}_0,\tilde{\mathbf{A}}+\nabla\tilde{\psi})). \quad (3.7)$$

Equations (3.3)–(3.7) are identical to those used in [10] except for the normalization. But our method of studying the instability is different from that used in this paper. We introduce comoving variables $(\mathbf{x}_\perp,\xi,t)$, the result being a set of linear partial differential equations the coefficients of which are periodic in ξ. Since the propagation of radiation in an unbounded plasma is considered we Fourier transform this set of equations in $\mathbf{x}_\perp$ and ξ,

$$(\tilde{\mathbf{A}},\tilde{\varphi},\tilde{n},\tilde{\psi})^T=(2\pi)^{-3/2}\int \exp[i((\mathbf{k},\mathbf{x}_\perp)+\chi\xi)] \times(\tilde{\mathbf{A}},\tilde{\varphi},\tilde{n},\tilde{\psi})^T(\mathbf{k},\chi,t)d^2\mathbf{k}d\chi.$$

The symbol ''$\sim$'' denoting perturbations is dropped in what follows for brevity. The resulting equations for the Fourier transforms are

$$\hat{D}A_1+g_1\frac{ik_1\chi^2}{\mathbf{k}^2+\chi^2}\psi-g_1\frac{k_1k_2}{\mathbf{k}^2+\chi^2}A_2+g_1\frac{k_2^2+\chi^2}{\mathbf{k}^2+\chi^2}A_1 +g_2[(F_{1,1}^-)_{\chi-k}+(F_{1,1}^+)_{\chi+k}] +\frac{g_1}{2}[(F_{1,2}^-)_{\chi-2k}+(F_{1,2}^+)_{\chi+2k}]=0, \quad (3.8)$$

$$\hat{D}A_2+g_1\frac{ik_2\chi^2}{\mathbf{k}^2+\chi^2}\psi-g_1\frac{k_1k_2}{\mathbf{k}^2+\chi^2}A_1+g_1\frac{k_1^2+\chi^2}{\mathbf{k}^2+\chi^2}A_2 +g_2[(F_{2,1}^-)_{\chi-k}+(F_{2,1}^+)_{\chi+k}] +\frac{g_1}{2}[(F_{2,2}^-)_{\chi-2k}+(F_{2,2}^+)_{\chi+2k}]=0, \quad (3.9)$$

$$-(\mathbf{k}^2+\chi^2)\varphi=n, \quad (3.10)$$

$$\hat{D}_t\psi-\varphi+g_2[(\Pi_1+i\Pi_2)_{\chi-k}+(\Pi_1-i\Pi_2)_{\chi+k}]=0, \quad (3.11)$$

$$\hat{D}_tn=(\gamma_0^{-1}\chi^2+(\gamma_0^{-1}-g_1)\mathbf{k}^2)\psi+ig_1(k_1A_1+k_2A_2) -g_2((ik_1-k_2)n_{\chi-k}+(ik_1+k_2)n_{\chi+k}) +\frac{g_1}{2}[(ik_1-k_2)\Pi_1-(ik_2+k_1)\Pi_2]_{\chi-2k} +\frac{g_1}{2}[(ik_1+k_2)\Pi_1-(ik_2-k_1)\Pi_2]_{\chi+2k}. \quad (3.12)$$

Here

$$g_1=\frac{a_0^2}{2\gamma_0^3},\quad g_2=\frac{a_0}{2\gamma_0},\quad \mathbf{k}^2=k_1^2+k_2^2,$$

$$\mathbf{A}=(A_1,A_2,A_3),\quad \Pi_{1,2}=A_{1,2}+ik_{1,2}\psi,$$

$$\hat{D}_t=\partial_t-iq\chi,\quad \hat{D}=-\partial_t^2+2iq\chi\partial_t+\left(\frac{\chi^2}{\gamma_0k^2}-\mathbf{k}^2\right)-\gamma_0^{-1},$$

$$F_{1,1}^{\pm}=\left(k_1\frac{k_1\mp ik_2}{\mathbf{k}^2+\chi^2}-1\right)n,$$

$$F_{1,2}^{\pm}=\left(1-k_1\frac{k_1\mp ik_2}{\mathbf{k}^2+\chi^2}\right)(\Pi_1\mp i\Pi_2),$$

$$F_{2,1}^{\pm}=-in\left(ik_2\frac{k_1\mp ik_2}{\mathbf{k}^2+\chi^2}\mp 1\right),$$

$$F_{2,2}^{\pm}=\left(k_2\frac{ik_1\pm k_2}{\mathbf{k}^2+\chi^2}\mp 1\right)(i\Pi_1\pm\Pi_2).$$

Shifting the argument χ in the above equations by $\pm lk$, where l stands for an integer, we establish an infinite set of coupled linear ordinary differential equations for the amplitudes of laser radiation harmonics comprising the perturbations. This set can be presented as

$$Y_t=BY, \quad (3.13)$$

where Y is an infinite column and B is an infinite 35-diagonal matrix. Below the eigenvalues of matrix B will be calculated numerically with the help of the techniques presented in [36].

IV. SLAB GEOMETRY EQUATIONS

In the slab geometry case $k_1=0$ and $k_2=0$, and Eqs. (3.8)–(3.12) become

$$\hat{D}_1A_1+g_1A_1-g_2(n_{\chi-k}+n_{\chi+k}) +\frac{g_1}{2}[(A_1+iA_2)_{\chi-2k}+(A_1-iA_2)_{\chi+2k}]=0, \quad (4.1)$$

$$\hat{D}_1 A_2 + g_1 A_2 + i g_2 (n_{\chi+k} - n_{\chi-k}) + \frac{g_1}{2}[(iA_1 - A_2)_{\chi-2k} - (iA_1 + A_2)_{\chi+2k}] = 0, \quad (4.2)$$

$$\hat{D}_t \psi + \chi^{-2} n + g_2 [(A_1 + iA_2)_{\chi-k} + (A_1 - iA_2)_{\chi+k}] = 0, \quad (4.3)$$

$$\hat{D}_t n - \gamma_0^{-1} \chi^2 \psi = 0, \quad (4.4)$$

where

$$\hat{D}_1 = -\partial_t^2 + 2iq\chi\partial_t + \left(\frac{\chi^2}{\gamma_0 k^2}\right) - \gamma_0^{-1}.$$

Their form and the structure of the corresponding matrix B in Eq. (3.13) are similar to those of the three-dimensional case.

V. CONSERVED CIRCULAR POLARIZATION APPROXIMATION

In contrast to [9,13], Eqs. (3.8)–(3.12) and their slab geometry analogs (4.1)–(4.4) describe electromagnetic field perturbations of arbitrary polarization. Circular polarization of perturbations was assumed in [9,13] resulting in a substantial loss of generality. Let us demonstrate that under certain assumptions Eqs. (4.1)–(4.4) are reduced to those for the circularly polarized laser pulses.

Consider the slab geometry case. For circularly polarized perturbations of the electromagnetic radiation we have

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{e}_1 + i\mathbf{e}_2) a \ \exp(i(kx_3 - \omega t)) + \text{c.c.},$$

where a is the slow amplitude of the electromagnetic field. In terms of Fourier transforms the above equation is written as

$$A_1 = \tfrac{1}{2}(a_{\chi-k} + b_{\chi+k}), \quad A_2 = \frac{i}{2}(a_{\chi-k} - b_{\chi+k}), \quad (5.1)$$

a_χ and b_χ being the Fourier transforms of a and a^*. Substituting these relations in Eqs. (4.1) and (4.2) we arrive at the following equation for a:

$$-a_{tt} + 2iq\chi a_t + \frac{\chi^2}{k^2\gamma_0} a + 2i\left(\omega a_t - \frac{i\chi}{k\gamma_0} a\right) = 2g_2 n - g_1(a+b). \quad (5.2)$$

Differentiating Eq. (4.4) in time, using Eq. (4.3), and expressing a_χ and b_χ from Eq. (5.1), we derive an equation for the plasma wake to the propagating radiation,

$$[(\partial_t - iq\chi)^2 + \gamma_0^{-1}] n = -\frac{a_0 \chi^2}{2\gamma_0^2}(a+b). \quad (5.3)$$

Equations (5.2) and (5.3) are the Fourier transforms of the equations used in [9] to study the instability of laser radiation in plasma in the framework of the conserved circular polarization assumption.

VI. NUMERICAL ANALYSIS OF MAXIMAL GROWTH RATES OF THE INSTABILITY OF CIRCULARLY POLARIZED LASER RADIATION IN PLASMAS

A. Slab geometry

Consider the growth rate for problems (4.1)–(4.4). The results of computations are depicted in Figs. 3 and 4. A range of values of dimensions of matrix B were considered according to: $l = 6 + 12j$, $j = 0,1,2, \ldots 17$. Higher values of j correspond to higher numbers of harmonics $2j+1$ included in the simulation, as seen in Fig. 3. Incident laser radiation frequency corresponds to $j=1$. The wave vector of the scattered radiation is χ. Propagation in the positive direction of the x_3 axis corresponds to $\chi > 0$ and propagation in the negative direction corresponds to $\chi < 0$.

Perturbation growth rates as functions of χ are shown in Figs. 3 and 4. The scattered radiation comprises a set of harmonics and each of them is a doublet consisting of a Stokes and an anti-Stokes component. Therefore all the peaks in Figs. 3 and 4 are located at $\chi = \pm jk \pm k_p$. There are also small peaks near $\chi = \pm jk$ corresponding to the fluid dynamics analog of Compton scattering.

Computations show that for sufficiently high values of m the changes in the growth rates become small in all the harmonics except for a few highest ones (see Fig. 3). In other words, there is a boundary effect in simulations with a finite dimension B matrix. For example, the $j = 0,1, \ldots 14$ harmonics are described well enough for $m = 210$ whereas the $j = 15,16,17$ harmonics are distorted by the boundary effect.

Figure 4 shows the simulation results corresponding to an electron concentration, which is a factor of 2 lower than in the case depicted in Fig. 3. Consequently the plasma frequency ω_p and the plasma wave vector k_p are a factor of $2^{1/2}$ lower and the Raman scattering components are located a factor of $2^{1/2}$ closer in the case depicted in Fig. 4.

As follows from Fig. 4, Raman scattering components are broader when the intensity of the incident laser pulse is higher. Pairs of Raman scattering components merge at $a_0 \cong 1$. Therefore harmonics components are distinguishable in the relativistic case ($a_0 > 1$) but the Raman scattering components can be undistinguishable.

Importantly, understanding the dependencies of the scattering bandwidths on the intensity of the propagating laser radiation can be used for the experimental diagnostics of the propagation process.

It is interesting to compare the results of the above simulations to those of [9] where circular polarization of scattered radiation was assumed in the case of slab geometry. As we have seen this assumption leads to not detecting the infinite set of harmonics. Then the dimension of the problem matrix is 6×6. But for an arbitrary polarization all the harmonics are excited and interact in the slab geometry case. Mathematically this amounts to the necessity to solve an infinite set of coupled linear differential equations. Note that the simulations performed with the B matrix of a low 8×8 dimension yield results similar to those of [9].

B. Three-dimensional case

In this section the growth rate of problems (3.8)–(3.12) is calculated as the function of the three components of the

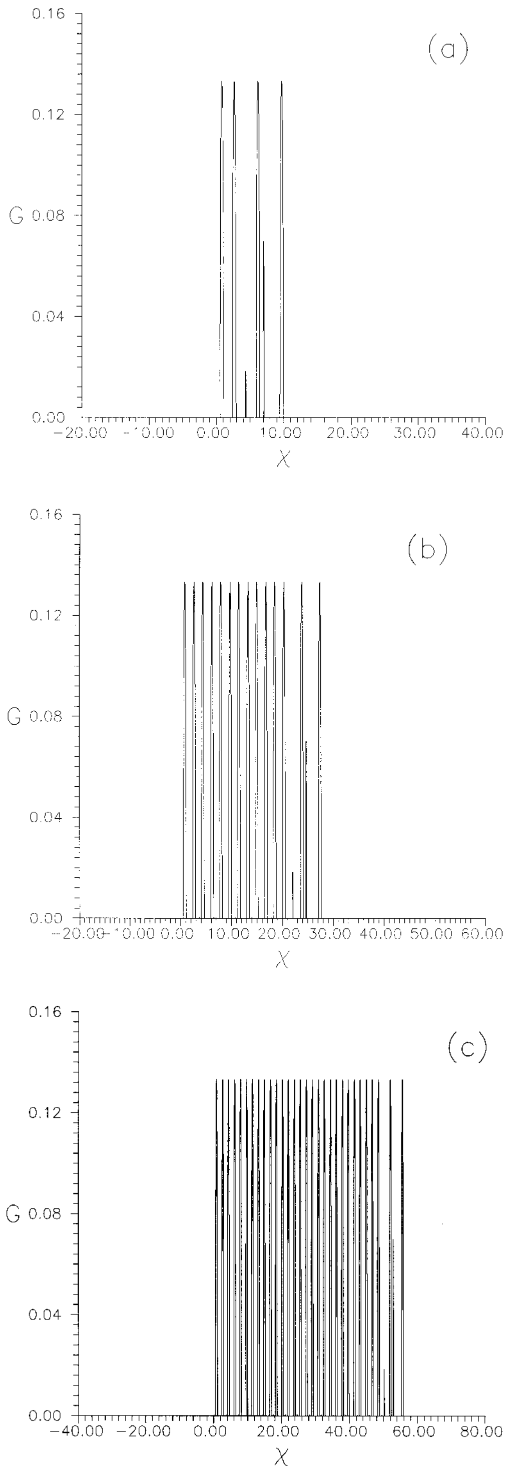

FIG. 3. Slab geometry. The influence of the number of harmonics included in the simulation on the growth rate. Number of harmonics: (a) $2j+1=5$, (b) 15, (c) 35. Parameter values are $a_0=0.1$, $\varepsilon^2=7.43\times10^{-2}$. Dependencies of the growth rate on the wave vector for $\chi>0$ are depicted. The plot is symmetric in χ.

wave vector $(\mathbf{k},\chi)$. However, only dependencies on two variables can be plotted. For example, we consider the following growth rate distributions: $G(k_1,0,\chi)$, $G(0,k_2,\chi)$, $G(k_1,k_2,0)$, and $G(k_\perp \cos\alpha, k_\perp \sin\alpha,\chi)$, where α is an angle and $k_\perp=\sqrt{k_1^2+k_2^2}$. Simulations show that the growth rate is quasiperiodic in χ and that its transverse distributions are not

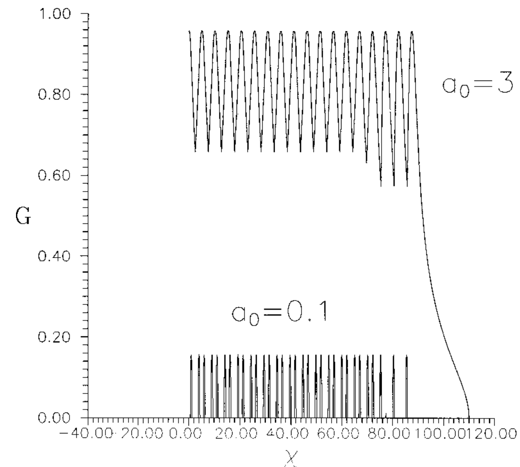

FIG. 4. Slab geometry. The dependence of the instability growth rate on the incident wave intensity for $\chi>0$. The plot is symmetric in χ. Incident wave amplitudes: $a_0=0.1$; $a_0=3$. $\varepsilon^2=3.72\times10^{-2}$.

axially symmetric. The latter fact follows from the axial asymmetry of the linearized equations on the incident wave period. Figures 5, 6, and 7 show the growth rate of the instability of a plane circularly polarized monochromatic wave. The dependence of the growth rate on k_2 and χ for $k_1=0$ is depicted in Fig. 5. Figure 6 shows the same growth rate as the function of k_2 and χ for $(k_2/k_1)=10^{-3}$. The case $k_2=0$ is illustrated by Fig. 7.

The oscillating coefficients of the linearized equations are written in a different way in different reference frames. In our case the growth rate is calculated assuming that the vector $\mathbf{A}_0$ is directed along the $\mathbf{e}_1$ axis at $x_3=0$ for $t=0$. But the choice of the zero moment of time within the period of the incident wave is random. This means that the growth rate must be averaged over the initial time moment within the wave period, which is equivalent to averaging the growth rate over the azimuthal angle. So the unaveraged results are intermediate and ''are not observed,'' whereas the averaged values correspond to the physically observable quantities.

Figure 8 shows the averaged growth rate as the function of $k_\perp$ and χ. It is quasiperiodic in χ. Obviously there are (i) a set of interconnected rings, (ii) repetitive peaks located

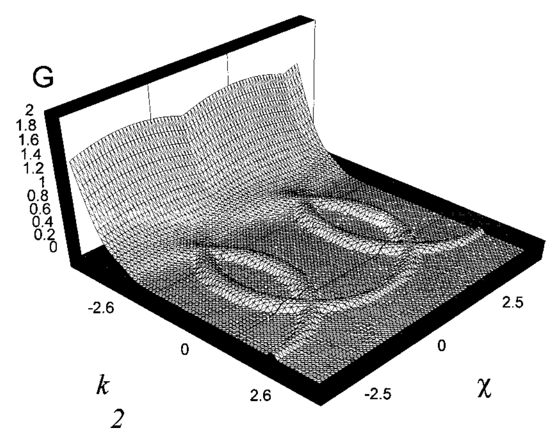

FIG. 5. The dependence of the instability growth rate on k_2 and χ for $k_1=0$. The simulation parameters are $a_0=0.1$, $\varepsilon^2=7.4362\times10^{-2}$.

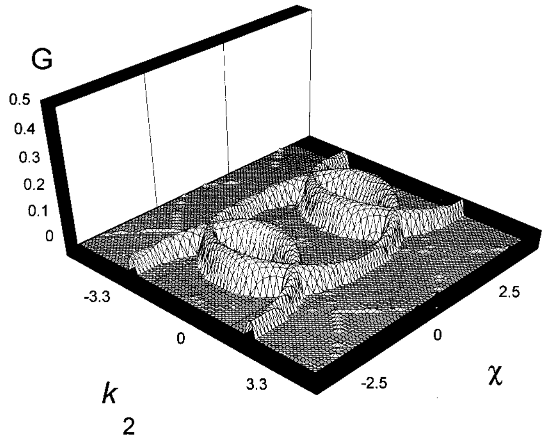

FIG. 6. The dependence of the instability growth rate on k_2 and χ for $(k_2/k_1)=10^{-3}$. The simulation parameters are $a_0=0.1$, $\varepsilon^2=7.4362\times10^{-2}$.

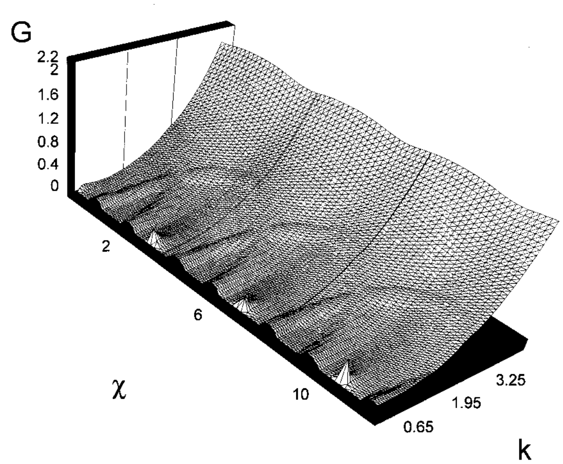

FIG. 8. The dependence of the two-dimensional growth rate distribution averaged over the azimuthal angle on $k_\perp$ and χ. The parameter values are $a_0=0.1$, $\varepsilon^2=7.43\times10^{-2}$.

near the $\mathbf{e}_3$ axis, and (iii) an increase in the growth rate for $k_\perp\rightarrow\infty$.

Details of the scattering of circularly polarized laser pulses in plasmas are illustrated in Fig. 8. Harmonics with wave vectors making $m\mathbf{k}_0-\delta\mathbf{k}$, $m=0,\pm1,\pm2,\ldots$ are excited in the plasma into which the incident pump wave propagates ($\mathbf{k}_0=\mathbf{e}_3k$ and $\delta\mathbf{k}$ is the wave-vector shift due to the electron response, $|\delta\mathbf{k}|\ll|\mathbf{k}_0|$). Energy and momentum are conserved by the initial set of equations [29]. Since no additional assumptions are introduced in their subsequent treatment the theory presented describes the electron response adequately. Due to the decay instability each of the harmonics gives rise to an electromagnetic wave (the Stokes component of Raman scattering) and a plasma wave:

$$m\mathbf{k}_0-\delta\mathbf{k}\rightarrow\mathbf{k}_m'+\mathbf{k}_e, \quad m\omega_0-\delta\omega=(m\omega_0-\delta\omega-\omega_p)+\omega_p.$$

Since the wave vector of cold plasma oscillations is arbitrary the direction of k_m' in space is also arbitrary. Thus the spatial distribution of the growth rate is similar to a circle with the radius making $|\mathbf{k}_m'|$. Furthermore, wave interactions result in scattered waves with $\mathbf{k}''=n\mathbf{k}_0+\mathbf{k}_m'$, where n and m are any integers. The structure of the $\mathbf{k}''=n\mathbf{k}_0+\mathbf{k}_1'$ rings can be seen in Fig. 8. The ring structures corresponding to large values of m are averaged out but they are found in unaveraged distributions. There are no anti-Stokes components in the medium for which the ground state involves no plasma wave with the frequency making ω_p, and the computations corroborate this fact. But wave interactions between the harmonics with $m\mathbf{k}_0$ and the backscattered Stokes components with $\mathbf{k}_n'$ do result in waves the wave vectors of which are directed along the propagation axis, their magnitudes making $(m-n)k+k_p$. Due to this the scattering pattern looks like there are anti-Stokes components in the slab geometry case. For this reason in the three-dimensional case anti-Stokes scattering components will be observed in narrow solid angles near the axis of the propagation of the incident wave.

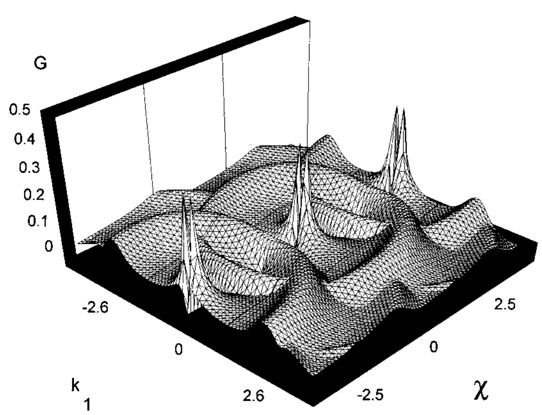

FIG. 7. The dependence of the instability growth rate on k_1 and χ for $k_2=0$. The simulation parameters are $a_0=0.1$, $\varepsilon^2=7.4362\times10^{-2}$.

The increase in the growth rate at $k_\perp\rightarrow\infty$ corresponds to the generation of a continuum of scattered radiation, i.e., to the emergence of radiation having a continuous spectrum. It is well known that synchrotron radiation is emitted by an electron traveling along a circular orbit and this radiation is an infinite set of harmonics [37]. When the circular orbits are distorted the radiation spectrum changes and the continuum emerges.

There are at least three reasons for the generation of a continuum in experiments: (1) Bremsstrahlung and, partially, photorecombination radiation in the plasma; (2) The laser radiation's being nonmonochromatic, which is of particular importance in the case of ultrashort pulses; (3) The nonharmonic character of the electron currents in plasma. This circumstance is related to the onset of plasma turbulence and the anomalous increase in the emission of radiation from it.

Eigenvectors of system (3.13) were calculated as well as growth rates. In the slab geometry case the eigenvector corresponding to the maximal growth rate is of a resonant character, i.e., it is not equal to zero for those values of χ, which are approximately equal to the wave vector of the harmonics. For those χ in the interval between the nearest two harmonics wave vectors only the components of the eigenvectors corresponding to these two harmonics are not equal to zero. It is of interest that in slab geometry the eigenvectors are found to be the superpositions of left and right circular polarizations.

A large number of eigenvector components are not equal to zero in the continuum area for the scattering at large angles to the propagation direction. There is little information in these eigenvectors since they cannot be normalized numerically.

The slab geometry problem analysis makes it possible to conclude that an initial perturbation that is localized in the **k** space evolves generating the nearest harmonics consequently.

The present theory leads to an interesting hypothesis on the polarization of the scattered radiation. Since at each moment of time perturbations in plasma infinitesimal volumes are sums of perturbations generated at different moments of time on the wave period (the problem is linear) and the asymptotic solutions corresponding to different initial moments of time differ from each other by a rotation of the **k** space by an azimuthal phase angle ψ around the $\mathbf{e}_3$ axis one should expect that the resulting vector potential average values satisfy the following relations: $\langle A_1 \rangle = 0$, $\langle A_2 \rangle = 0$, and $\langle A_3 \rangle \neq 0$. Naturally the average values of the squares of all these quantities are not equal to zero. Thus the radiation should be partially depolarized. A more detailed study of these issues should be based on the methods of statistical physics [38] and is beyond the scope of the present paper.

The theory presented above is applicable in the nonrelativistic limit as well. In this case the terms proportional to a_0^2 should be dropped and $\gamma_0 = 1$ should be assumed in Eqs. (3.3)–(3.7). In contrast to the relativistic case the corresponding eigenvalue problem is posed for a 20-diagonal matrix. The nonrelativistic simulations for $a_0 = 0.1$ yield results which are identical to those illustrated by Fig. 4(a).

The nonrelativistic problems have been examined for a long time and the approaches used were (a) the study of instabilities with the help of linearized equations and resonance approximations based on exact phase matching [24]; (b) the treatment of the dispersion relations without phase matching conditions [18] (initial and boundary-value problems). As follows from the present study resonance approximations should be avoided in the relativistic case due to the broadening of the resonant structures.

It is of interest to compare the results of the present work and the conclusions of [10], where dispersion relations for the scattering of a circularly polarized wave in the three-dimensional geometry were derived. The authors of [10] restricted their study to the case of $k_2 = 0$ assuming that the problem is axially symmetric (in fact the situation is more complicated and the growth rate must be averaged over the azimuthal angle in **k** space). A series of the Stokes harmonics of Raman scattering [10] corresponds to our set of rings for the scattered waves having the wave vector $\mathbf{k}'_m$. But there are the following differences: (a) in [10] the growth rate tends to zero when the polar angle tends to 0 and π, while in our unaveraged picture it tends to the values obtained by solving the slab geometry problem; (b) only the Stokes component of Raman scattering with the wave vector of $\mathbf{k}'_1$ on the continuum background is found in the averaged picture at large angles; (c) there are both Stokes and anti-Stokes components for all the harmonics scattered at small angles.

VII. CONCLUSIONS

The results of a rigorous linear three-dimensional instability analysis for the propagation of plane monochromatic circularly polarized waves in plasmas are presented. The following phenomena are described by the theory: (i) excitation of harmonics of the propagating laser radiation in the nonlinear medium; (ii) scattering by plasmons; (iii) scattering due to the fluid dynamics analog of Compton effect; (iv) decay instability at harmonics resulting in scattered electromagnetic waves and plasmons; (v) interactions of electromagnetic waves in plasma; (vi) generation of a continuum of scattered radiation.

The above phenomena are studied in both the relativistic and nonrelativistic cases.

Simulations show that scattering both forward and backward is possible. Radiation is comprised of a set of harmonics. The harmonics are scattered into a set of spatial cones imbedded in one another. Higher harmonics are scattered into more narrow cones. The spectrum of radiation scattered outside of the cones is continuous. The latter phenomenon is dominant.

Harmonics with lower numbers that are scattered into relatively wide cones can propagate outside the spatial area in which the incident laser pulse is localized whereas the high number harmonics propagate with the pulse. It should be possible to record them experimentally by the pulse spectral analysis using a specially arranged geometry. The intensity of scattering backwards is low due to the short time of interaction between counterpropagating beams [13].

A slab geometry theory of scattering of relativistically intense laser radiation in plasmas without assuming conservation of polarization of the incident laser pulse is a particular case of the general theory proposed in the present paper. It is shown that in this case scattering results in the excitation of a sequence of harmonics. Every harmonics constitutes a doublet consisting of a Stokes and an anti-Stokes component. The relativistic and charge-displacement nonlinearities are the mechanisms of harmonics excitation.

It is established in the framework of the three-dimensional theory that the most significant effects associated with scattering at large angles to the propagation direction are the generation of a Stokes component at the laser pulse frequency and of a continuum of scattered radiation. These phenomena can be observed experimentally.

The harmonics resulting from scattering in small solid angles include both Stokes and anti-Stokes components, the latter resulting from the interactions of the higher-order harmonics and backscattered Stokes components. It is shown that the harmonics are doublets in case the incident laser pulse intensity is nonrelativistic but components become broader and merge in the case of relativistically intense incident laser radiation. The latter fact can be helpful in experimental studies of laser-plasma interactions since the dependence of the scattering bandwidths on the laser intensity can be used in diagnostics of the propagation process.

The predictions of the three-dimensional theory for the scattering angles equal to 0 and π are identical to the results of the slab geometry simulations.

The technique of calculating the maximal growth rates numerically as the eigenvalues of the matrix of the right-hand sides of an infinite set of coupled ordinary differential equations is used for the first time for the treated class of problems. This approach is more efficient than deriving and investigating extremely complicated dispersion relations for the growth rates.

The following facts are important as far as the comparison of the proposed theoretical results with experimental data is concerned. First, absorption of laser radiation in plasma is neglected in the calculations. Including absorption in the model can alter the results slightly. Second, particular experimental geometry and the impact of a finite pulse duration can be important. The theory presented above is applicable if (a) the minimal scattering domain (the beam transverse size) is much greater than the wavelength; (b) the time of the development of the instability considered is less than the pulse duration.

Linearly polarized laser radiation is typically used in experiments with relativistic intensities whereas the theory is developed in the present paper for a circularly polarized ''ground state.'' However, the range of physical phenomena outlined above should occur in the case of linear polarization of laser pulses as well. It should be noted that a new theory of linearly polarized nonmonochromatic plane electromagnetic waves in plasmas was presented above.

ACKNOWLEDGMENT

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[1] P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. A **41**, 4463 (1990).
[2] P. Sprangle and E. Esarey, Phys. Rev. Lett. **67**, 2021 (1991).
[3] U. Johann *et al.*, Phys. Rev. A **34**, 1084 (1986).
[4] D. Umstadter, W. B. Mori, and C. Joshi, Phys. Fluids B **1**, 183 (1989).
[5] S. H. Batha, D. D. Meyerhofer, A. Simon, and R. P. Drake, Phys. Fluids B **3**, 448 (1991).
[6] K. Estabrook, W. L. Kruer, and M. G. Haines, Phys. Fluids B **1**, 1282 (1989).
[7] S. V. Bulanov *et al.*, Phys. Fluids B **4**, 1935 (1992).
[8] V. M. Gorbunov and V. I. Kirsanov, Tr. Fiz. Inst. im. P.N. Lebedeva, Ross. Akad. Nauk **219**, 3 (1992).
[9] V. I. Kirsanov and A. S. Sakharov, Fiz. Plazmy **21**, 623 (1995).
[10] V. I. Kirsanov and A. S. Sakharov, Fiz. Plazmy **21**, 632 (1995).
[11] A. S. Sakharov and V. I. Kirsanov, Phys. Plasmas **4**, 3382 (1997).
[12] A. V. Borovsky and O. B. Shiryaev, Zh. Eksp. Teor. Fiz. **110**, 865 (1996) [Sov. Phys. JETP **83**, 475 (1996)].
[13] A. V. Borovsky, A. L. Galkin, V. V. Korobkin, and O. B. Shiryaev, Kvant. Elektron. **24**, 929 (1997) [Sov. J. Quantum Electron. **27**, 903 (1997)].
[14] A. B. Borisov *et al.*, Phys. Rev. Lett. **68**, 2309 (1992).
[15] G. Mourou and D. Umstadter, Phys. Fluids B **4**, 2315 (1992).
[16] F. G. Patterson, R. Gonzales, and M. D. Perry, Opt. Lett. **16**, 1107 (1991).
[17] K. Krushelnick *et al.*, Phys. Rev. Lett. **75**, 3681 (1995).
[18] V. P. Silin, *Parametricheskoe Vozdeystvie Izlucheniya Bol'shoy Moschnosti na Plasmu* (Parametric Effect of High Power Radiation on Plasma) (Nauka, Moscow, 1973) (in Russian).
[19] N. Bloembergen, *Nonlinear Optics* (Addison-Wesley, Redwood City, CA, 1992).
[20] J. R. Lalanne, A. Ducasse, S. Kielich, *Laser Molecular Interactions: Laser Physics and Molecular Nonlinear Optics* (Wiley, New York, 1996).
[21] N. E. Andreev, Zh. Eksp. Teor. Fiz. **59**, 2105 (1970).
[22] L. M. Gorbunov, Usp. Fiz. Nauk **16**, 217 (1973).
[23] S. Jorna, Phys. Fluids **17**, 765 (1974).
[24] V. N. Tsytovich, *Theory of Turbulent Plasma* (Consultants Bureau, New York, 1977).
[25] C. J. McKinstrie, A. Simon, and E. A. Williams, Phys. Fluids **27**, 2738 (1984).
[26] R. L. Berger, Phys. Fluids **27**, 1796 (1984).
[27] A. I. Akhiezer and R. V. Polovin, Zh. Eksp. Teor. Fiz. **30**, 915 (1956) [Sov. Phys. JETP **3**, 696 (1956)].
[28] V. G. Priymak, Ph.D. thesis, Mathematical Modeling Institute, Moscow, 1996 (in Russian).
[29] A. V. Borovsky and A. L. Galkin, *Lazernaya Fizika* (IzdAT, Moscow, 1996) (in Russian).
[30] C. Max and F. W. Perkins, Phys. Rev. Lett. **27**, 1342 (1971).
[31] A. C. L. Chian and P. C. Clemmow, J. Plasma Phys. **14**, 505 (1975).
[32] P. K. Kaw, A. Sen, and E. J. Valeo, Physica D **9**, 96 (1983).
[33] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems* (Springer-Verlag, Berlin, 1990).
[34] R. N. Sudan, Y. S. Dimant, and O. B. Shiryaev, Phys. Plasmas **4**, 1489 (1997).
[35] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1960).
[36] J. H. Wilkinson and S. Reinsch, *Linear Algebra* (Springer-Verlag, Berlin, 1971).
[37] L. D. Landau and E. M. Lifshitz, *The Classical Field Theory* (Pergamon, Oxford, 1962).
[38] Yu. L. Klimontovich, *Statistical Physics* (Harwood Academic, New York, 1986).